Question Sheet 8, Differentiation III. v1 2020-21

Taylor Polynomials

If f has n derivatives at $a \in \mathbb{R}$ then

$$T_{n,a} f\left(x\right) = \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!} \left(x-a\right)^{r}.$$

There are four questions asking you to calculating Taylor polynomials and they all highlight a method that should simplify the work needed and cut down the opportunity of making an error.

1. Calculate the Taylor polynomial
$$T_{6,0}\left(\frac{\sin x + \cos x}{1+x}\right).$$
Hint Multiply up so you don't have to differentiate rational functions.

2. Calculate the Taylor polynomial
$$T_{8,0}\left(\sin x \cosh x\right).$$
Hint Look for a pattern in your derivatives. For the trigonometric functions $\sin x$ and $\cos x$ you return to a function related to the original function after differentiation at most 4 times. For hyperbolic functions it is after 2 differentiations. Thus for f that are products of such functions you might hope to see some connection between f and $f^{(4)}$.

3. Calculate the Taylor polynomial
$$T_{5,0}\left(e^{\sin x}\right).$$
Hint Let $f(x) = e^{\sin x}$ and, because of the exponential function satisfies $de^x/dx = e^x$, look for a connection between f and $f^{(1)}$.

4. Calculate the Taylor Polynomial
$$T_{4,0}\left(\frac{\ln\left(1+x\right)}{1+x}\right).$$
Hint Again look at multiplying up and writing a derivative in terms of earlier derivatives.

Error Terms

The Remainder or Error Term in approximating a function by it's Taylor Polynomial is given by

$$R_{n,a}f(x) = f(x) - T_{n,a}f(x)\,.$$

In the notes we give bounds on $R_{n,a}f(x)$ which thus tell us how well $T_{n,a}f(x)$ approximates $f(x)$. This is the subject of the next three questions. But we can also deduce something from knowing that $R_{n,a}f(x)$ is of constant sign as x varies; we get inequalities between $f(x)$ and $T_{n,a}f(x)$.

5. i. Prove that

$$\left|\sin x - x + \frac{x^3}{6}\right| \le \frac{1}{4!}\left|x\right|^4, \tag{1}$$

for all $x \in \mathbb{R}$.

Hint the left hand side is $|R_{3,0}(\sin x)|$.

ii. Deduce (without L'Hôpital's Rule) that

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

6. For $f(x) = \ln(1+x)$, find the Taylor polynomial $T_{5,0}f(x)$ and calculate $T_{5,0}f(0.2)$.

Use Lagrange's form of the error for the remainder to estimate the error in using $T_{5,0}f(0.2)$ to calculate $\ln 1.2$.

Hence show that

$$0.18232000... < \ln 1.2 < 0.18232709...\ .$$

7. Use Taylor's Theorem with $f(x) = \sqrt{x}$ on $[64, 66]$ and $n = 1$ along with *Lagrange*'s form of the error to show that

$$\frac{1}{8} - \frac{1}{1024} < \sqrt{66} - 8 < \frac{1}{8} - \frac{1}{1458}.$$

Taylor Series

8. Calculate the Taylor Series for $x \cosh x + \sinh x$ with $a = 0$.

9. Prove that the Taylor series for cosine converges to $\cos x$, i.e.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!},$$

for all $x \in \mathbb{R}$.

Additional Questions

10. Assume the function f is $n + 1$ times differentiable with $f^{(n+1)}$ continuous on an open interval containing $a \in \mathbb{R}$. Prove that

$$\lim_{x \to a} \frac{f(x) - T_{n,a}f(x)}{(x-a)^n} = 0 \quad \text{and} \quad \lim_{x \to a} \frac{f(x) - T_{n,a}f(x)}{(x-a)^{n+1}} = \frac{f^{(n+1)}(a)}{(n+1)!}. \tag{2}$$

Hint Consider Lagrange's error.

Note these limits in special cases have been seen many times before.

(a) $f(x) = \sin x$ with $T_{2,0}(\sin x) = x$ is the subject of Question 5,

(b) $f(x) = e^x$ with $T_{3,0}(e^x) = 1 + x + x^2/2$ is the subject of Question 9 on Sheet 3.

(c) $f(x) = \sinh x$ with $T_{2,0}(\sinh x) = x$ is the subject of the same question. To check that earlier answer

$$\begin{aligned} \lim_{x \to 0} \frac{\sinh x - x}{x^3} &= \lim_{x \to 0} \frac{\sinh x - T_{2,0}(\sinh x)}{x^3} \\ &= \frac{1}{3!} \frac{d^3}{dx^3} (\sinh x) \bigg|_{x=0} \quad \text{by (2)} \\ &= \frac{1}{6}. \end{aligned}$$

11. i. Prove that $x^{n+1}R_{n,0}(e^x) \geq 0$ for all $x \in \mathbb{R}$.

Deduce that for all $m \geq 1$ we have

$$e^x \geq T_{2m-1,0}(e^x)$$

for all $x \in \mathbb{R}$, while

$$\begin{cases} e^x \geq T_{2m,0}(e^x) & \text{for } x > 0 \\ e^x \leq T_{2m,0}(e^x) & \text{for } x < 0. \end{cases}$$

Note this answers a question in the printed lecture notes, of showing that

$$e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

for all $x \in \mathbb{R}$ while

$$e^x > 1 + x + \frac{x^2}{2} \text{ if } x > 0 \quad \text{and} \quad e^x < 1 + x + \frac{x^2}{2} \text{ if } x < 0.$$

ii. Prove that $(-1)^n x^{n+1}R_{n,0}(\ln(1+x)) \geq 0$ for all $x > -1$.

Deduce that for all $n \geq 1$,

$$\ln(1+x) \leq T_{n,0}(\ln(1+x))$$

for $-1 < x < 0$ while if $x > 0$ then

$$\begin{cases} \ln(1+x) \leq T_{n,0}(\ln(1+x)) & \text{for odd } n \\ \ln(1+x) \geq T_{n,0}(\ln(1+x)) & \text{for even } n. \end{cases}$$

Note These last results for $x > 0$ can be combined in

$$T_{2m,0}(\ln(1+x)) \leq \ln(1+x) \leq T_{2m+1,0}(\ln(1+x))$$

for all $m \geq 1$. The case $m = 1$ is the content of Question 6, Sheet 7.